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TOPICAL ISSUE

Hierarchical Games and Computational Procedures in the Linear Case

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Abstract—The prerequisites for the emergence of the theory of hierarchical games are described. The models are described and linked to a set of articles that give an idea of the key problems and the fundamental results obtained. Attention is paid to practical applications. The basic concepts and provisions are illustrated by solutions in the case of linear problems. In conclusion, the need for further research in the field of computational methods is noted.

Keywords: operations research, theory, hierarchical games, results, practical applications, linear problems, computational methods

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1. INTRODUCTION

The discipline of operations research in the domestic sense arose on the basis of developments on the efficiency of various controlled processes [1]. A characteristic feature of domestic developments was the fundamental saturation of productions with mathematical models.

In this paper, the authors continue these traditions, making sure that all new approaches fit quite well into the design of operations research.

The principle of optimality adopted in the domestic school is based on the idea of guaranteed result when analyzing operations containing uncertainties. Much attention is paid to the order of moves of the active participants in the operation, which made it possible to consider a wide class of problems with a hierarchical structure.

Significant progress has been made in the development of numerical methods for solving operations research problems using penalty function methods.

The paper presents further progress in the above-mentioned areas.

2. HIERARCHY. EXAMPLES OF DEVELOPMENTS

Issues of control and decision-making in organizational systems with hierarchical structure attracted much attention and have significant literature.

Let us dwell, in particular, on some examples from foreign publications.

The 2016 Nobel Prize in Economics was awarded to Oliver Hart (Harvard University, USA) and Bengt Holmström (Massachusetts Institute of Technology, USA) for their contributions to contract theory (where there is a leader and followers: Principal and agents), which is based on models of interaction with a hierarchical nature. The official announcement of the award states: "The modern economy contains an innumerable number of contracts. New theoretical tools, created by the authors (Hart and Holmström), are valuable for understanding real contracts and institutions, as well as to take into account possible pitfalls in the development of contracts. Their optimal contract analysis lays the intellectual foundation for the development of strategies and institutions in many areas, from legislation on bankruptcy to political constitutions" [2]. Contract theory has become an integral part of modern economic theory.

In foreign literature, when modeling hierarchical systems, Stackelberg equilibrium models are most often used. Stackelberg's main works were published in Austria in the 1930s. Noting the hierarchical nature of the structure of the economy, he wrote: "...why should we nationalize production, it would be better for us to nationalize the owners of the enterprises."

Stackelberg's ideas became more widely known in the 1950s, when his books were translated into English [3].

Fundamental works on the analysis of conflict situations in hierarchical systems were obtained by the school of Germeier Yu.B., a significant part is summarized in [4].

The hierarchy is seen as a natural development of the principles of optimality in game theory and natural application to operations research problems, where the consideration is carried out in the interests of the operating party.

The formulation of the problem of analyzing hierarchical games has attracted considerable interest from a large group of researchers at the Computing Center of the Federal Research Center of the Russian Academy of Sciences (Computing Center of the USSR Academy of Sciences) and the Faculty of Computational Mathematics and Cybernetics of Moscow State University. The activity demonstrated resulted in three monographs and several dozen articles in leading journals, presentations at conferences, and applied developments in various spheres [5–11].

The work of the school attracted the attention of A.B. Rapoport, a leading researcher in systems theory, former visiting director of the Institute for Advanced Studies in Vienna, and A.B. Rappoport made and published the English translation [12]. In the preface to the translation of the book, A.B. Rapoport writes: "Beyond the Soviet Union analysis of this kind has focused primarily on price-setting processes, where the basic structures of the game are, for the most part, quite elementary. In the work of Germeier and his students, the structure in which the process takes place (which includes much more, than trade), was significantly expanded and diversified. Especially an interesting topic, one to which Soviet game theorists devoted considerable attention, is the topic of the so-called hierarchical games, which can be interpreted as models of planned economies with varying degrees of centralization or decentralization."

It seems that this assessment is positive on the one hand, and on the other hand played a negative role since it shifted the emphasis of opinions of Western fellow economists, having tied the theory of hierarchical games too much to the image of a planned economy, which has a negative character for them. This is unfounded since the principle of a guaranteed result does not exclude equilibrium, which is remarkably illustrated by the example of the saddle point existence theorem in games with opposing interests.

Closely related to the current direction of operations research are the open control theory, a review of which is published in [13], and the theory of mechanisms [14].

An example of an application can be control in large-scale projects of a multi-structured economy, where full information support is required for making strategic decisions [15, 16].

At present, these trends are acquiring specific organizational forms: "Significant advances in the automation of decision support systems in organizational systems have been achieved within the framework of the System of Distributed Situation Centers (SDSC), which are considered the technological and hardware-software basis of decision support systems. The main goal of modern situation centers is to support decision-making processes by control based on visual representations (images) of situations arising in the controlled environment, to provide control with a visualization of the results of their analysis in the most convenient form for decision-making. Not only analysis, but also forecasting, trends in the development of the situation in the short, medium, and long term" [17].

However, these needs were recognized quite a long time ago. 50 years ago, academician V.M. Glushkov wrote: "We should also note that in domestic developments of the past, which summarized the ideas of control of economic systems by that time, the question of the new significance of information in the life of society was fundamentally raised and a three-level computer system in the territorial aspect in a human-machine version was described, which, accumulating and processing information, would generate draft state plans and implement decision-making functions. The system was called the General State Automated Control System (OGAS)." [18].

3. HIERARCHICAL GAMES THEORY MODELS

The emergence of new information technologies, such as artificial intelligence, forces us to take a new look at some problems of the theory of hierarchical systems. Many economists, both theorists and practicing businessmen, draw attention to the relevance of this kind of research.

Recently [19, 20] a formal analogy has been established between the problem of finding weights in deep learning of artificial neural networks and the problem of analyzing the interaction of active elements of a multi-level hierarchical system, striving to optimize some common criterion. Probably, the analysis of this analogy can be useful in solving both problems.

In the theory of operations research [1], the position is accepted that "an operation is a set of purposeful actions." The mathematical model of an operation typically includes phase variables, uncertain factors, and control variables that generate the corresponding actions.

In classical models of hierarchical game theory [4, 8], the following assumptions are made:

- The system has a dedicated element (the Center), whose interests are identified with the interests of the system as a whole.
- Center has the right of the first move, i.e., it is the first to choose its strategy, and this choice becomes known to its counterparties.
- Center's strategy may have a complex functional structure if the Center expects to receive some information about the actions of its partners.
- It is assumed that the subordinate elements of the system act rationally, based on their own interests within the framework of the rules determined by the Center.
- Center is cautious, i.e., it is guided by the worst for its choice of partners.

From a theoretical point of view, it should be noted that the study should combine simulation and optimization approaches.

4. DESCRIPTION OF TECHNOLOGICAL LIMITATIONS

In this paper, linear models of controlled systems are adopted as the object of study. The choice of the nature of the dependence is determined by the substantive and analytical possibilities provided by the fundamental theories of balances by V.V. Leontiev and production processes by Kantorovich and Koopmans.

The term "agent" is understood in the sense of the definition from the work [21]: "a typical subject whose interests and awareness correspond to the role he plays in a given social system of division of labor is called an agent. Note that in society the system of roles is structured in such a way that the coordination of interests with roles occurs at all levels of the hierarchy, therefore, an agent can be understood not only as an individual but also as a legal entity — an organization."

Let's consider a linear model of control of a complex production system.

We will assume that the system under consideration consists of n agents. We will number them with natural numbers from 1 to n.

Let the system produce m types of products, and let the production process use k different types of resources. Let us assume that in the process to produce one unit of product of type j, the ith agent spends resource of type l in the amount of p^i_{lj}. The system has centralized reserves of resources of type l in the amount of r_l $(l = 1, \ldots, k)$. In addition, the ith agent has its own reserve of such a resource. Its volume is denoted by b^i_l.

It is assumed that the output of type j can be priced at c_j.

We will assume that the cost standards are independent of the volume of production. Under this assumption, when producing x^i_j units of output of type j $(j = 1, \ldots, m)$ by agent i is spent $p^i_{l1}x^i_1 + p^i_{l2}x^i_2 + \ldots + p^i_{lm}x^i_m$ units of resource of type l. For all produced goods, the agent i can earn the sum $c_1x^i_1 + c_2x^i_2 + \ldots + c_mx^i_m$.

We will consider the prices c_j to be positive. It goes without saying that the cost coefficients p^i_{lj} are non-negative. No output can be produced without any costs. Therefore, it is natural to assume that for all i and all j at least one coefficient $p^i_{1j}, \ldots, p^i_{kj}$ is strictly positive. It is reasonable to think the volumes of reserves b^i_l and r_l non-negative.

To shorten the formulas, we use matrix notations. The output vector of the ith agent $(x^i_1, \ldots, x^i_m)^{\mathrm{T}}$ will be denoted by the symbol x^i (superscript "T" denotes transposition). Let $c = (c_1, \ldots, c_m)$ be the price vector, and

$$P^i = \begin{pmatrix} p^i_{11} & p^i_{12} & \cdots & p^i_{1m} \\ p^i_{21} & p^i_{22} & \cdots & p^i_{2m} \\ \cdots & \cdots & \cdots & \cdots \\ p^i_{k1} & p^i_{k2} & \cdots & p^i_{km} \end{pmatrix}$$

—cost matrix. Then the cost vector $y^i = \left(y^i_1, \ldots, y^i_k\right)^{\mathrm{T}}$ satisfies the condition $y^i = P^ix^i$, and the amount received by agent i for the released product is equal to cx^i.

For the stock vectors, we introduce the notation $b^i = \left(b^i_1, \ldots, b^i_k\right)^{\mathrm{T}}$, $r = (r_1, \ldots, r_k)^{\mathrm{T}}$.

Let Y denote the set of all sets of vectors $y^1, y^2, \ldots, y^n$, satisfying the conditions

$$y^1 + \ldots + y^n \leqslant r,$$
$$y^i \geqslant 0, \quad i = 1, 2, \ldots, n$$

(as usual, all vector inequalities are understood as component-wise).

As noted above, the issue of studying hierarchical systems is closely related to issues of centralization and decentralization [22, 23].

For illustrative purposes, we will give a limited consideration of some of the problems, applications of theoretical research of economic operations in organizational systems.

5. CENTRALIZED DISTRIBUTION OF RESOURCES

In this section, we will assume that there is a certain subject (Center) that controls the activities of all agents. Namely, the Center determines the volumes of centralized resources y^i allocated to each of the agents and their output volumes x^i. The goal of control will be considered to be maximization of the total income of all agents.

Let us assume that the parameters of the system related to individual agents are not known exactly to the Center. Such a situation can be modeled in various ways. In this case, we will use the "interval" method. We will assume that the Center does not know the exact norms p^i_{lj}

of the supplies and the "own" stocks of the agents, but it knows only that $p_{lj}^{i-} \leqslant p_{lj}^{i} \leqslant p_{lj}^{i+}$ and $b_l^{i-} \leqslant b_l^i \leqslant b_l^{i+}$.

Having defined the matrices

$$P^{i-} = \begin{pmatrix} p_{11}^{i-} & p_{12}^{i-} & \dots & p_{1m}^{i-} \\ p_{21}^{i-} & p_{22}^{i-} & \dots & p_{2m}^{i-} \\ \dots & \dots & \dots & \dots \\ p_{k1}^{i-} & p_{k2}^{i-} & \dots & p_{km}^{i-} \end{pmatrix}, \quad P^{i+} = \begin{pmatrix} p_{11}^{i+} & p_{12}^{i+} & \dots & p_{1m}^{i+} \\ p_{21}^{i+} & p_{22}^{i+} & \dots & p_{2m}^{i+} \\ \dots & \dots & \dots & \dots \\ p_{k1}^{i+} & p_{k2}^{i+} & \dots & p_{km}^{i+} \end{pmatrix},$$

and vectors

$$b^{i-} = \left(b_1^{i-}, \dots, b_k^{i-}\right)^{\mathrm{T}}, \quad b^{i+} = \left(b_1^{i+}, \dots, b_k^{i+}\right)^{\mathrm{T}},$$

these conditions can be written more compactly as $P^{i-} \leqslant P^i \leqslant P^{i+}$ and $b^{i-} \leqslant b^i \leqslant b^{i+}$. Of course, it is assumed that the matrices P^{i-}, P^{i+} and the vectors b^{i-}, b^{i+} are known to the Center.

It is natural to assume that the Center cannot violate the laws of conservation of "physical" quantities, therefore, it must choose its controls in such a way that each agent has enough resources for its own production program for any values of uncertain parameters (from given intervals).

Thus, we obtain a rather unusual optimization problem:

$$cx^1 + cx^2 + \ldots + cx^n \to \max,$$

subject to constraints

$$P^i x^i \leqslant b^i + y^i, \quad i = 1, \dots, n,$$
$$y^1 + y^2 + \ldots + y^n \leqslant r, \quad x^i \geqslant 0, \quad y^i \geqslant 0, \quad i = 1, \dots, n.$$

Its unusualness lies in the fact that the restrictions $P^i x^i \leqslant b^i + y^i$, $i = 1, 2, \dots, n$, must be satisfied for any $P^{i-} \leqslant P^i \leqslant P^{i+}$ and $b^{i-} \leqslant b^i \leqslant b^{i+}$. Thus, formally, we obtain a problem with an infinite number of constraints. However, this problem is easily solved.

Indeed, if the inequalities $P^{i+} x^i \leqslant b^{i-} + y^i$, $i = 1, \dots, n$, are satisfied, then the inequalities $P^i x^i \leqslant b^i + y^i$, $i = 1, \dots, n$, will be satisfied for all $P^i \leqslant P^{i+}$ and $b^{i-} \leqslant b^i$. Of course, the condition $P^{i+} x^i \leqslant b^{i-} + y^i$, $i = 1, \dots, n$ is also necessary.

This means that the problem posed is equivalent to the standard linear programming problem

$$cx^1 + cx^2 + \ldots + cx^n \to \max,$$
$$P^{i+} x^i \leqslant b^{i-} + y^i, \quad i = 1, \dots, n,$$
$$y^1 + y^2 + \ldots + y^n \leqslant r, \quad x^i \geqslant 0, \quad y^i \geqslant 0, \quad i = 1, \dots, n,$$

controls in which are the output vectors x^i and vectors of resources allocated to agents y^i $(i = 1, \dots, n)$.

The additional structure of this problem, associated with the presence of several agents, allows its decomposition using the ideology of Lagrange multipliers.

According to the Kuhn–Tucker theorem, there exists a vector $\lambda = (\lambda_1, \dots, \lambda_k)$ with non-negative components, that one of the points of maximum of the function

$$cx^1 + cx^2 + \ldots + cx^n - \lambda y^1 - \lambda y^2 - \ldots - \lambda y^n$$

under restrictions

$$P^{i+} x^i \leqslant b^{i-} + y^i, \; i = 1, \dots, n,$$
$$x^i \geqslant 0, \; y^i \geqslant 0, \; i = 1, \dots, n$$

is a solution to the linear programming problem under consideration.

To find this maximum point, n "small" linear programming problems can be solved

$$cx^i - \lambda y^i \to \max$$
$$P^{i+}x^i \leqslant b^{i-} + y^i,\ x^i \geqslant 0,\ \ y^i \geqslant 0$$

with controls x^i and y^i.

This circumstance can be used to construct iterative procedures for finding solutions to the main problem. But in this article, the interpretation of these constructions is more important.

The number λ_l can be interpreted as the price of a resource of type l. This price appoints the Center. After that, each agent chooses their own release program x^i and the volume of purchased resources y^i in order to maximize its profit $cx^i - \lambda y^i$. If the agents are friendly to the Center, then with such a decentralized choice of control, the "general" constraint $y^1 + \ldots + y^n \leqslant r$ will be satisfied. Otherwise, in this case, it is impossible to solve the problem by choosing prices alone, and it is necessary to provide for some other mechanisms for coordinating the actions of agents.

Remark 1. The situation here is quite typical for linear models. Let there be several identical enterprises capable of producing two types of products, trying to maximize their own profit. If the prices of products are such that the first type of product is more profitable than the second, then no one will produce the second type of product. Therefore, if the task is to produce a given set of products, then these prices must be such that the profit from the production of both types of products is the same. But then all the program production of any enterprise, fully loading its capacities, will be optimal. And with independent decision-making, there are no guarantees that the right set of products will be produced. Therefore, some mechanism for coordinating decisions is needed. However, there are usually no particular problems here, since this mechanism will dictate to each enterprise a choice of equally advantageous solutions.

Remark 2. If we take an admissible point (x^i, y^i) of last problem, then for any positive t the point (tx^i, ty^i) will also be admissible. But when multiplying variables by t, the criterion value will also be multiplied by t. Consequently, the optimal value of the criterion is 0. Therefore, it makes sense to change the sign of the criterion and talk about minimizing losses rather than maximizing profits.

Remark 3. It is clear that for positive y^i_l the constraint

$$p^i_{l1}x^i_1 + p^i_{l2}x^i_2 + \ldots + p^i_{lm}x^i_m \leqslant b^i_l + y^i_l$$

at the optimum point turns into equality. Therefore, for $k > m$ the solution to the last problem will be degenerate. It is very likely that when solving the problem using the simplex method, degeneration will occur still "on the way" to the optimum.

6. DECENTRALIZED PRODUCTION WITH INDEPENDENT AGENTS

The interpretation described in Section 5 leads to the consideration of a different control scheme for the same system.

Let there still be a dedicated party (the Center) that has the right to choose the distribution of "common" resources $y^1, \ldots, y^n$. Its goal is to maximize total production, estimated using the price vector c. The Center makes its choice first, and this choice becomes known to all agents.

Based on the resources available to him, the ith agent chooses the output volume x^i. In doing so, he seeks to maximize the cost of released products $c^i x^i$.

We will still assume that the Center knows the limits within which the coefficients of the cost matrices P^i and vectors of stocks b^i $(P^{i-} \leqslant P^i \leqslant P^{i+}$ and $b^{i-} \leqslant b^i \leqslant b^{i+})$. In addition, the Center knows exactly the interests of all agents (vectors $c^i = (c^i_1, \ldots, c^i_m)$).

Let us introduce some notation. The set of matrices P^i satisfying the conditions $P^{i-} \leqslant P^i \leqslant P^{i+}$ we denote by Π^i, and a set of vectors b^i satisfying the constraints $b^{i-} \leqslant b^i \leqslant b^{i+}$ – through B^i.

It is natural to assume that agent i knows exactly "its" matrix P^i and vector b^i.

Under the described assumptions, each agent solves a simple optimization problem

$$c^i x^i \to \max, P^i x^i \leqslant b^i + y^i, x^i \geqslant 0,$$

and, what is especially important, the Center, knowing the interests of the agents, is able to correctly estimate the set $BR^i(y^i, P^i, b^i)$ of all solutions to this problem. If he counts on the rational behavior of all agents and is cautious about the remaining uncertainty, it must focus on the maximal guaranteed result

$$\sup_{(y^1,y^2,\ldots,y^n)\in Y} \min_{(P^1,P^2,\ldots,P^n)\in\Pi^1\times\Pi^2\times\ldots\times\Pi^n} \min_{(b^1,b^2,\ldots,b^n)\in B^1\times B^2\times\ldots\times B^n}$$
$$\min_{(x^1,x^2,\ldots,x^n)\in BR^1(y^1,P^1,b^1)\times BR^2(y^2,P^2,b^2)\times\ldots\times BR^n(y^n,P^n,b^n)} \left(cx^1 + cx^2 + \ldots + cx^n\right).$$

Let's try to correlate this result with the Center's result in the problem from Section 5 (in fact, two control schemes are considered there, but they give the same results).

Let us fix an arbitrary optimal solution $y^1, \ldots, y^n, x^1, \ldots, x^n$ tasks from Section 5 and arbitrary distribution of $v^1, \ldots, v^n$ resources, satisfying the constraints

$$v^1 + v^2 + \ldots + v^n \leqslant r, v^i \geqslant 0,\ i = 1, \ldots, n.$$

Let z^i be the optimal solution to the problem

$$cz^i \to \max, P^{i+} z^i \leqslant b^{i-} + v^i, z^i \geqslant 0.$$

Then for any w^i satisfying the conditions $P^{i+} w^i \leqslant b^{i-} + v^i$, $w^i \geqslant 0$, the inequality $cw^i \leqslant cz^i$ holds. In particular, the inequality $cw^i \leqslant cz^i$ holds for any $w^i \in BR(v^i, P^{i+}, b^{i-})$.

Adding these inequalities, we get $cw^1 + cw^2 + \ldots + cw^n \leqslant cz^1 + cz^2 + \ldots + cz^n$.

As shown in Section 5, for the solution of the problem considered there, the following restrictions are satisfied:

$$P^{i+} x^i \leqslant b^{i-} + y^i,\ i = 1, 2, \ldots, n, y^1 + y^2 + \ldots + y^n \leqslant r, x^i \geqslant 0,\ y^i \geqslant 0,\ i = 1, \ldots, n.$$

Therefore, $cz^1 + cz^2 + \ldots + cz^n \leqslant cx^1 + cx^2 + \ldots + cx^n$.

Then the inequality $cw^1 + cw^2 + \ldots + cw^n \leqslant cx^1 + cx^2 + \ldots + cx^n$ is true.

Especially

$$\min_{(w^1,w^2,\ldots,w^n)\in BR^1(w^1,P^1,b^1)\times BR^2(w^2,P^2,b^2)\times\ldots\times BR^n(w^n,P^n,b^n)} \left(cw^1 + cw^2 + \ldots + cw^n\right)$$
$$\leqslant cx^1 + cx^2 + \ldots + cx^n$$

and

$$\min_{(P^1,P^2,\ldots,P^n)\in\Pi^1\times\Pi^2\times\ldots\times\Pi^n} \min_{(b^1,b^2,\ldots,b^n)\in B^1\times B^2\times\ldots\times B^n}$$
$$\min_{(w^1,w^2,\ldots,w^n)\in BR^1(w^1,P^1,b^1)\times BR^2(w^2,P^2,b^2)\times\ldots\times BR^n(w^n,P^n,b^n)} \left(cw^1 + cw^2 + \ldots + cw^n\right)$$
$$\leqslant cx^1 + cx^2 + \ldots + cx^n.$$

Since the distribution of $v^1, v^2, \ldots, v^n$ was chosen arbitrarily, it follows that

$$\sup_{(v^1,v^2,\ldots,v^n)\in Y}\ \min_{(P^1,P^2,\ldots,P^n)\in\Pi^1\times\Pi^2\times\ldots\times\Pi^n}\ \min_{(b^1,b^2,\ldots,b^n)\in B^1\times B^2\times\ldots\times B^n}$$
$$\min_{(w^1,w^2,\ldots,w^n)\in BR^1(w^1,P^1,b^1)\times BR^2(w^2,P^2,b^2)\times\ldots\times BR^n(w^n,P^n,b^n)}\left(cw^1+cw^2+\ldots+cw^n\right)$$
$$\leqslant cx^1+cx^2+\ldots+cx^n.$$

Thus, the maximal guaranteed result of the Center in the problem from this section always does not exceed the maximal guaranteed result of the Center in the problem from Section 5. An example of a problem (without interpretation) in which this inequality is strict can be constructed without much difficulty.

It has been known for a long time that in conditions of absence of uncertainty, the two considered methods of control are equally effective. Here a new qualitative effect appears: under uncertainty, the "economic" method of control (using resource prices) may turn out to be more effective than the directive (using the choice of physical indicators). This effect has yet to be studied in detail.

7. FAN STRUCTURE. TWO LEVELS

In [24, 25] it is noted that the above formulated problems about distribution of some resource by the Center can be written in a two-level system in the general form: to find

$$\max_{u\in D}\left(\min_{x\in T(u)}\sum_{j=1}^{n}k_jx_j\right)=\max_{u\in D}F(u),$$

where

$$T(u)=\left\{x|x\in T_0(u),\sum_{j=1}^{n}c_jx_j=\max_{y\in T_0(u)}\sum_{j=1}^{n}c_jy_j\right\},$$
$$T_0(u)=\left\{x|x\in E^n,x\geqslant 0,\sum_{j=1}^{n}a_{ij}x_j=b_i+\sum_{l=1}^{k}b_{il}u_l,i=1,\ldots,m\right\},$$
$$D=\left\{u|u\in E^n,u\geqslant 0,\sum_{l=1}^{k}d_{rl}u_l\leqslant d_r,r=1,\ldots,p\right\}.$$

Problems in this formulation have been considered by many authors; the present statement follows the formulation from [24].

Let us introduce the function

$$F_0(u)=\sum_{j=1}^{u}k_jx_j,$$

where $x\in T(u)$. This function cannot be defined for all values of $u\in\mathbb{R}^k$, since $T(u)$ may be the empty set. In particular, if for all values of u the formulated problem has no solutions, then $F_0(u)$ is not defined at all. Further, we will consider that there is $u_0\in D$ such that $T(u_0)\neq\emptyset$ and $T(u_0)$ is a bounded set. If $T(u)$ contains more than one element, then the function $F_0(u)$ can take several values. We also introduce the notation

$$F(u)=\min_{x\in T(u)}F_0(u).$$

The following statement is true, which allowed us to move from the maximin tasks for optimization.

Theorem 1 [24]. *There exists $\delta_0 > 0$ such that for all $\delta,\ 0 < \delta < \delta_0$, the function*

$$F_0^\delta = \sum_{j=1}^{n} k_j x_j,$$

where $x \in T^\delta(u)$, is single-valued and $F(u) = F_0^\delta(u)$,

$$T^\delta(u) = \left\{ x | x \in T_0(u), \sum_{j=1}^{n} (c_j - \delta k_j) x_j = \max_{y \in T_0(u)} \sum_{j=1}^{n} (c_j - \delta k_j) y_j \right\}.$$

Remark 4. When constructing an algorithm, the specific value of δ_0 that is defined in the theorem is not used. Only the fact of the existence of such a quantity is important.

8. FAN STRUCTURE. THREE-LEVEL SYSTEM

Let us present a simplified version of the strategic planning model [20], taking into account the interaction of the active elements of the system. The purpose of constructing this model is to compare different control schemes.

The prototype for this model was a system consisting of a Center, several integrated structures (holdings), and a set of enterprises included in these holdings. There are technological links between the elements of such a system. Enterprises, using their capacities and the resources allocated to them, produce products and transfer them to the next level (to "their" holdings). Holdings process them and transfer them to the Center. Organizational links are imposed on this structure: higher-level elements determine the amounts of resources allocated to their subordinates. All these relationships are naturally modeled by a game with a hierarchical structure.

Let us consider a model of a three-level hierarchical system consisting of a Center and two agents.

The center selects a column vector x from the set

$$X = \{x : x \geqslant 0, Ax \leqslant a\}.$$

Here A is a matrix with non-negative coefficients, a is a column vector with positive elements. We will additionally assume that each column of matrix A contains at least one strictly positive element (no type products are not produced without the expenditure of at least some resources). Matrix A and vector a are the task parameters.

Under the assumptions made, the set X is a non-empty closed convex bounded polyhedron.

Remark 5. We can consider models in which a is a vector with non-negative elements. This case is easily reduced to the one considered with a reduction in task dimensionality.

Given a vector x, the top-level agent selects a column vector y from sets

$$Y(x) = \{y : y \geqslant 0, By \leqslant x\}.$$

Here B is a given matrix with non-negative elements that does not contain zero columns.

It is directly verified that for any x the set $X(y)$ is non-empty convex and compact polyhedron.

Finally, if a vector y is selected, the lower-level agent selects a vector z from the set

$$Z(y) = \{z : z \geqslant 0, Cz \leqslant y\}.$$

The matrix C with non-negative elements, not containing zero columns, is again assumed to be a task parameter.

From the assumptions made, it follows that the set $Z(y)$ is a non-empty closed convex and bounded polyhedron.

The goal of the lower-level agent is to maximize the value of the function $h(z) = rz$, where r – the given row vector. Similarly, the interests of the top-level agent are described as the desire to increase the value of the function $g(y) = qy$, where the row vector q is considered a parameter of the problem. Finally, we will assume that the Center's gain is determined by function $f(z) = pz$, where p is a given row vector. The elements of the vectors p, q, r will be considered non-negative.

The dimensions of the vectors a, x, y and z are assumed to be finite and, generally speaking, not coinciding. We will assume that the dimensions of the matrices A, B, C and the vectors p, q, r are such that all the formulas written above are formally correct (thus, these dimensions are determined uniquely).

We assume that all parameters of the system, i.e., matrices A, B, C and vectors p, q, r are exactly known to the Center.

The following decision-making procedure is proposed. First, the Center selects its strategy x from the set X. Then the top-level agent chooses its strategy y from the set $Y(x)$. And finally, the lower-level agent chooses strategy z from sets $Z(y)$.

Given the assumptions made, it is natural for the top-level agent to choose its strategy from the set

$$BR^t(x) = \left\{ y \in Y(x) : g(y) = \max_{v \in Y(x)} g(v) \right\}.$$

Similarly, all reasonable choices of the lower-level agent and only these belong to the set

$$BR^l(y) = \left\{ z \in Z(y) : h(z) = \max_{w \in Z(y)} h(w) \right\}.$$

Under the assumptions made, for any $x \in X$ the maximum in definition of the set $BR^t(x)$ is achieved, and the set $BR^t(x)$ itself is a non-empty convex and compact polyhedron. Similarly, for any vector y with non-negative components, the set $BR^l(y)$ is non-empty convex closed and bounded polyhedron.

If the Center knows the parameters of the model, it is able to independently evaluate sets $BR^t(x)$ and $BR^l(y)$. The Center cannot more accurately estimate the set of possible choices of agents. If the Center is cautious, then its maximal guaranteed result is

$$\sup_{x \in X} \min_{y \in BR^t(x)} \min_{z \in BR^l(y)} f(z).$$

We will deal with the problem of calculating this value.

From the above, it follows that the minimums in the last formula are achieved. The attainability of the upper bound will have to be investigated separately.

9. EQUIVALENT PROBLEM

The constraints $y \in BR^t(x)$ and $z \in BR^l(y)$ are not standard: they contain (previously unknown) maxima of the functions. At least for some purposes, these limitations should be eliminated, even at the cost of increasing the dimension of the problem. Let's do it.

Let $x \in X$ be given. For $v \in Y(x)$ and $w \in Z(y)$ we define the sets

$$V(x, v) = \{ y \in Y(x) : qy \geqslant qv \},$$
$$W(y, w) = \{ z \in Z(y) : rz \geqslant rw \}.$$

For all x, y, v, w the sets $V(x, v)$ and $W(y, w)$ are compact convex polyhedra.

For any $w \in Z(y)$ and any $z \in BR^l(y)$ the inequality $rz \geqslant rw$ holds, i.e., the inclusion $BR^l(y) \subseteq W(y, w)$ is valid. Therefore,

$$\min_{z \in BR^l(y)} f(z) \geqslant \min_{z \in W(y,w)} f(z),$$

and due to the arbitrariness of $w \in Z(y)$ the inequality

$$\min_{z \in BR^l(y)} f(z) \geqslant \sup_{w \in Z(y)} \min_{z \in W(y,w)} f(z)$$

holds.

Let's choose now $w' \in Z(y)$ so that $w' \in BR^l(y)$. Then by definition we have $W(y, w') = BR^l(y)$. Hence,

$$\min_{z \in BR^l(y)} f(z) = \min_{z \in W(y,w')} f(z)$$

and even more so

$$\min_{z \in BR^l(y)} f(z) \leqslant \sup_{w \in Z(y)} \min_{z \in W(y,w)} f(z).$$

Therefore, the upper bound in the last formula is achieved, and

$$\min_{z \in BR^l(y)} f(z) = \max_{w \in Z(y)} \min_{z \in W(y,w)} f(z).$$

Similarly, for any $v \in Y(x)$ and any $y \in BR^t(x)$ the inequality $qy \geqslant qv$, holds. i.e., the inclusion $BR^t(x) \subseteq V(x, v)$ is valid. Therefore,

$$\min_{y \in BR^t(x)} \max_{w \in Z(y)} \min_{z \in W(y,w)} f(z) \geqslant \min_{y \in V(x,v)} \max_{w \in Z(y)} \min_{z \in W(y,w)} f(z)$$

and due to the arbitrariness of $v \in Y(x)$ the inequality

$$\min_{y \in BR^t(x)} \max_{w \in Z(y)} \min_{z \in W(y,w)} f(z) \geqslant \sup_{v \in Y(x)} \min_{y \in V(x,v)} \max_{w \in Z(y)} \min_{z \in W(y,w)} f(z)$$

holds.

Let's choose now $v' \in Y(x)$ so that $v' \in BR^t(x)$. Then $V(x, v') = BR^t(x)$ and that's why

$$\min_{y \in BR^t(x)} \max_{w \in Z(y)} \min_{z \in W(y,w)} f(z) = \min_{y \in V(x,v')} \max_{w \in Z(y)} \min_{z \in W(y,w)} f(z).$$

Especially

$$\min_{y \in BR^t(x)} \max_{w \in Z(y)} \min_{z \in W(y,w)} f(z) \leqslant \sup_{v \in Y(x)} \min_{y \in V(x,v)} \max_{w \in Z(y)} \min_{z \in W(y,w)} f(z).$$

This means that the upper bound in the last formula is achieved and the equalities

$$\min_{y \in BR^t(x)} \max_{w \in Z(y)} \min_{z \in W(y,w)} f(z) = \max_{v \in Y(x)} \min_{y \in V(x,v)} \max_{w \in Z(y)} \min_{z \in W(y,w)} f(z),$$

or

$$\min_{y \in BR^t(x)} \min_{z \in BR^l(y)} f(z) = \max_{v \in Y(x)} \min_{y \in V(x,v)} \max_{w \in Z(y)} \min_{z \in W(y,w)} f(z)$$

are satisfied.

Due to the arbitrariness of $x \in X$ we get from here

$$\sup_{x \in X} \min_{y \in BR^t(x)} \min_{z \in BR^l(y)} f(z) = \sup_{x \in X} \max_{v \in Y(x)} \min_{y \in V(x,v)} \max_{w \in Z(y)} \min_{z \in W(y,w)} f(z).$$

The problems of calculating the maximal guaranteed result in such cases are extremely complex. Therefore, it seems that the presence of two different calculation methods would be useful, even if neither of them is very effective on its own.

In addition, the obtained result allows us to more adequately assess the complexity of the problem being solved.

10. EXISTENCE OF A SOLUTION

Let us formulate some preliminary results.

Let $x \in \mathbb{R}$. The system of linear inequalities $Ax \leqslant a$ defines a certain set of points $P = \{x : Ax \leqslant a\}$. If this set is not empty, then it is a polyhedron. For simplicity, we will assume that this set is not empty and bounded. Then it is compact.

Let a^i $(i = 1, \ldots, k)$ be the rows of the matrix A, and a_i be the components of the column vector a. For points $x \in P$ we define a set $I(x)$ of indices $i = 1, 2, \ldots, d$, for which $a^i x = a_i$. Let $J(x) = \{1, , k\} \setminus I(x)$.

Let $I \subset \{1, \ldots, k\}$. If the set $P_I = \{x \in P : I(x) = I\}$ is not empty, then it is a face of the polyhedron P. If the set I is such that the system of vectors $a^i,\ i \in I$, contains d linearly independent vectors, then the set P_I is either empty or consists of one point. In the latter case, this point is the vertex of the polyhedron P.

Let L_I be the linear span of the set P_I. Then the face P_I is an open subset of the space L_I (in the topology induced by the Euclidean topology on $\mathbb{R}^d$), since it is defined by a system of strict inequalities $a^i x < a_i,\ i \in J(y)$ for some point $y \in P_I$.

The polyhedron P is the union of its pairwise non-intersecting faces.

Lemma 1. *The closure of any face P_I of a polyhedron P contains at least one vertex of this polyhedron.*

The proof of the lemma is contained in the Appendix.

In the future, the following version of the Lagrange multiplier principle will be used.

Lemma 2. *Let $x \in P$. Point x is the maximum of function px on set P if and only if there exist non-negative numbers $\lambda^i,\ i \in I(x)$, such that $p = \sum_{i \in I(x)} \lambda^i a^i$.*

The proof of the lemma is given in the Appendix.

Now let's get back to the main task.

The outcome (x, y, z) will be called semi-optimal if $y \in BR^t(x)$, $z \in BR^l(y)$ and $f(z) = \min\limits_{z' \in BR^l(y)} f(z')$. By definition, the maximal guaranteed result of the Center is equal to the exact upper bound of the function $f(z)$ by the set of all semi-optimal outcomes (x, y, z).

Let's consider a polyhedron

$$P = \{(x, y, z) : x \geqslant 0, Ax \leqslant a, y \geqslant 0, By \leqslant x, z \geqslant 0, Cz \leqslant y\}.$$

Let the outcome $(x, y, z) \in P$ be semi-optimal. We will show that then every outcome (x', y', z'), belonging to the face $P_{I(x,y,z)}$ of the polyhedron P, containing the point (x, y, z), is semi-optimal.

Let's consider polyhedra

$$\Pi(x) = \{(y, z) : y \geqslant 0, By \leqslant x, z \geqslant 0, Cz \leqslant y\}$$

and

$$\Pi(x') = \{(y, z) : y \geqslant 0, By \leqslant x', z \geqslant 0, Cz \leqslant y\}.$$

We denote $\bar{x} = (y, z)$, $\bar{x}' = (y', z')$, and the inequalities defining the polyhedron $\Pi(x)$ we write it in the form $\bar{A}\bar{x} \leqslant \bar{a}(x)$, where $\bar{A}$ is some matrix, and $\bar{a}(x)$ is some vector. Then the polyhedron $\Pi(x')$ will be defined by the inequalities $\bar{A}\bar{x} \leqslant \bar{a}(x')$ with the same matrix $\bar{A}$ and possibly a different vector $\bar{a}(x')$.

Let $\bar{a}^i$ be the rows of matrix $\bar{A}$, $\bar{a}_i(x)$ and $\bar{a}_i(x')$ be elements of the vectors $\bar{a}(x)$ and $\bar{a}(x')$, respectively, and $I(\bar{x})$ is the set of all indices i for which the equalities $\bar{a}^i\bar{x} = \bar{a}_i(x)$ are satisfied. Since the outcomes (x, y, z) and (x', y', z') belong to the same face of the polyhedron P, for the same indices the equalities $\bar{a}^i\bar{x}' = \bar{a}_i(x')$ are satisfied.

By virtue of the necessary condition of Lemma 2 and the inclusion $y \in BR^t(x)$ there exist such non-negative numbers λ^i, $i \in I(x)$, such that $q = \sum_{i \in I(\bar{x})} \lambda^i \bar{a}^i$. But then, by virtue of the sufficient condition of Lemma 2, the inclusion $y' \in BR^t(x')$ holds.

Finally, let's consider polyhedra

$$\Pi'(y) = \{z : z \geqslant 0, Cz \leqslant y\}$$

and

$$\Pi'(y') = \{z : z \geqslant 0, Cz \leqslant y'\}.$$

Repeating the same reasoning using these polyhedra, we will see that the inclusion $z' \in BR^l(y')$ is valid.

Let's now consider polyhedra

$$\Pi''(y, z) = \{z'' : z'' \geqslant 0, Cz'' \leqslant y, rz'' = rz\}$$

and

$$\Pi''(y', z') = \{z'' : z'' \geqslant 0, Cz'' \leqslant y', rz'' = rz'\}.$$

Repeating the same reasoning again, but using the polyhedra $\Pi''(y, z)$ and $\Pi''(y', z')$, we arrive at the equality $f(z') = \min_{z'' \in BR^l(y')} f(z'')$.

Thus, the outcome (x', y', z') is semi-optimal.

By continuity, if the outcome (x, y, z), is semi-optimal, then all outcomes from the closure of the face $P_{I(x,y,z)}$ of the polyhedron P, containing the point (x, y, z) will be semi-optimal.

But the closure of any face of a polyhedron is itself a closed polyhedron. Therefore, the maximum of the linear function $f(z) = pz$ on the closure of any face is necessarily achieved at one of the vertices of the polyhedron. Therefore, for any of the semi-optimal outcome (x, y, z) there exists a vertex (x', y', z'), which is the semi-optimal outcome, and in addition, $pz' \geqslant pz$.

And since the set of vertices of the polyhedron P is finite, the upper bound

$$\sup_{x \in X} \min_{y \in BR^t(x)} \min_{z \in BR^l(y)} f(z)$$

is certainly achieved, and at such a point x, that there exist such $y \in Y(x)$ and $z \in Z(y)$, for which the triple (x, y, z) is a vertex of the polyhedron P.

Testing for semi-optimality is a procedure comparable in complexity to one step of the simplex method. Therefore, the obtained results reduce the problem under consideration to enumeration of all the vertices of the polyhedron P.

In general, it is probably impossible to avoid such an exhaustive search. Therefore, analyzing a truly serious model in this way is hardly possible. But the study of truly serious models is usually carried out in several stages, the first of which is the study of a simplified model. At this stage, it will most likely be possible to obtain an exact solution to the problem using enumeration. And then one can use some heuristic methods (like the branch and bound method), taking into account the specifics of a particular model.

11. CONCLUSION

The paper presents further advances in the applications of hierarchical games in modeling controlled processes based on linear dependencies.

The results obtained will undoubtedly be included in the toolkit of operations research and formalized approaches in decision-making theory.

It was noted above that the problem of analyzing a linear hierarchical game is reduced to enumerating the vertices of a certain polyhedron, the number of which may be quite large. Therefore, for the numerical solution of the formulated problems, it is necessary to involve various approximate methods, which will be the subject of further developments.

APPENDIX

Proof of Lemma 1. If P_I is a vertex, then everything is obvious. Otherwise the set P_I is open (in L_I), and its closure is closed. Therefore, there exists a point y belonging to the closure but not belonging to the face P_I itself. Let $x \in P_I$.

Then $I(x) \subseteq I(y)$. Indeed, if the equality $a^i x = a_i$, $i \in I(x)$, is true, then for any $z \in P_I$ the equality $a^i z = a_i$ is also true. And then, by continuity, the equality $a^i y = a_i$ is valid. Since $y \notin I$, the inclusion $I(x) \subseteq I(y)$ is strict.

In fact, the set $I(y)$ contains an index j such that the vector a^j is not linearly dependent on the vectors a^i, $i \in I(x)$. Indeed, let j belong to $I(y)$, but not belongs to $I(x)$. Suppose the vector a^j is linearly expressed through the vectors a^i, $i \in I(x)$. Then from the equalities $a^i x = a_i$ follows the equality $a^j x = a_j$, which contradicts the condition $j \notin I(x)$.

The face $P_I(y)$ belongs to the closure of the face P_I. The same reasoning can be applied to the face $P_I(y)$, further expanding the set $I(y)$. But since the set $1, 2, \ldots, k$ is finite, such a procedure cannot continue indefinitely. So, at some point we will come to face, which is the vertex. It will be the one we are looking for.

The lemma is proven.

Proof of Lemma 2. *Necessity.* By virtue of the Kuhn–Tucker theorem, there exist non-negative numbers λ^i, for which $p = \sum_{i=1}^{k} \lambda^i a^i$. On the strength of the conditions of complementary slackness $\lambda^i = 0$ for $i \in J(x)$. Therefore, $p = \sum\limits_{i \in I(x)} \lambda^i a^i$.

Sufficiency. Let y be an arbitrary point of the set P. Then

$$py = \sum_{i \in I(x)} \lambda^i a^i y \leqslant \sum_{i \in I(x)} \lambda^i a_i = \sum_{i \in I(x)} \lambda^i a^i x = px.$$

By definition, this means that x is a maximum point.

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